Nonlinear Wave Propagation

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14. ABSTRACT
The Principal Investigators research program in nonlinear wave propagation with emphasis in nonlinear optics is very active and covers a number of research areas. During the period 15 April, 2012 -- 14 April, 2015, fourteen papers were published in refereed journals. In addition, two refereed conference proceeding were published, and twenty one invited lectures were given. The key results and research directions are described in the associated Final Report. Full details can be found in the Principal Investigators research papers which are listed at the end of the Report. Key research investigations and areas studied include the following. A detailed theory describing the dynamics of nonlinear waves in photonic honeycomb lattices both in bulk regions and with boundaries was developed.

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OBJECTIVES

To carry out fundamental and wide ranging research investigations involving the nonlinear wave propagation which arise in physically significant systems with emphasis on nonlinear optics. The modeling and computational studies of wave phenomena in nonlinear optics include nonlinear wave propagation in photonic lattices, ultrashort pulse dynamics in mode-locked lasers, dynamics of dark solitons and analysis of dispersive shock waves.

STATUS OF EFFORT

The PI's research program in nonlinear wave propagation with emphasis in nonlinear optics is extremely active and covers a number of research areas. There have been a number of important research contributions carried out as part of the effort . During the period 15 April, 2012 – 14 April, 2015, fourteen papers were published in refereed journals. In addition, two refereed conference proceeding were published, and twenty one invited lectures were given. The PI also received a number of honors during this period.

The key results and research directions are described below in the section on accomplishments/new findings. Full details can be found in our research papers which are listed at the end of this report.

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The lattice nonlinear Schrödinger equation, which is derived from Maxwell's equations under the paraxial approximation, provides the starting point of the analysis associated with lattice photonics. The linear index of refraction is associated with the lattice background; the nonlinearity is derived from the inclusion of cubic nonlinearity. Often the background potential is periodic, but not always. In our research we have also studied quasi-crystal lattices and lattices with defects; we have carried detailed computational studies investigating stability properties, boundaries of associated bandgaps with solitons and vortices.

When the Schrödiunger potential is taken large, one can employ an approximation, called the tight binding limit. Within this approximation key nonlinear discrete systems are derived. In a regime near 'Dirac points', associated with special locations in the underlying linear dispersion relation, which exhibit conical singular structure, new phenomena are found. Some of the properties can be described in terms of the continuous limit where special (nonlinear Dirac type) systems are obtained. When the background lattice is weak different types of Dirac systems result.

Our first research efforts describe the wave dynamics in situations where boundary effects were not important. Subsequently we investigated cases when boundary edges and effects are important. In the latter case novel dynamics are fund to occur. In such situations sometimes unidirectional waves associated with Floquet topological insulators can be found. Such waves can propagate over very long distances with no backscatter. This situation is under current careful study.

Besides studying the dynamics and propagation of edge wave dynamics in honeycomb lattices we have also analyzed the propagation of multidimensional nonlinear surface plasmon waves in dielectric/metal interfaces.

Novel continuous and discrete systems closely related to the classical nonlinear Schrdinger equation and which exhibit parity time symmetry were solved exactly via the inverse scattering transform. They have have the property that the nonlinear induced potential simultaneously depends on current and reflected spatial locations.

The properties of mode-locked lasers which are used to create ultrashort pulses were analyzed

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under conditions where one has higher order perturbative terms such as third order and nonlinear dispersion. The basic mathematical model employs gain and filtering terms saturated by energy and a loss term saturated by power; this is referred to as the power energy saturation (PES) equation. Since dark solitons in mode locked lasers have been observed, we also used this model to study their mode locking characteristics.

Motivated by experiments at the University of Colorado, dispersive shock waves (DSWs) were also investigated. DSWs are found in many applications including nonlinear optics, fluid dynamics and Bose Einstein condensation. We showed that for certain fundamental nonlinear wave equations with multi-step initial conditions, even though the data breaks up in to numerous DSWs in an intermediate long time limit, eventually the solution tends to one DSW.

Nonlinear photonic lattices

Nonlinear light wave propagation in periodic optical photonic lattices is an exciting and active area of research. This is partially due to the realization that photonic lattices can be constructed on extremely small scales of only a few microns in size. Hence they allow the possibility of careful manipulation and navigation of lightwaves on small scales. Localized nonlinear optical pulses have been studied when they occur on one and two dimensional backgrounds. In one dimension the periodic backgrounds are often referred to as waveguide arrays

The periodic backgrounds can be either fabricated mechanically such as those comprised of AlGaAs materials (cf. [1] where one dimensional arrays are considered) or all-optically using photo-refractive materials as done in [2, 3]. These latter photonic structures can be constructed via interference of two or more plane waves.

Waveguide arrays

One dimensional coupled optical waveguides is a setting that represents a convenient laboratory for experimental observations. By etching the surface of the optical material one can form a periodic structure which is referred to here as a waveguide array. Typically, such a waveguide array is composed of numerous single-mode individual waveguides each being a few microns in width and a few millimeters in length. Such small scale structures could be embedded in a large scale environment and potentially can be used to guide light in a controllable manner.

In general an input optical beam will diffract in the lateral or horizontal direction unless it is balanced by nonlinearity. The first theoretical prediction of nonlinear wave propagation in an optical waveguide array which possessed localized discrete waves; i.e. discrete solitons, was reported in [4]. The wave phenomena satisfies the well known discrete nonlinear Schrödinger (DNLS) equation

$$i\frac{\partial E_n}{\partial z} + C(E_{n+1} + E_{n-1}) + \kappa|E_n|^2 E_n = 0 \,,$$

where E_n is the field strength at site n, C is a coupling constant which depends on adjacent waveg-

uides, κ is a constant describing the nonlinear index change and z is the direction of propagation.

In 1998 self-trapping of light in a discrete nonlinear waveguide array was experimentally observed by a group of researchers at the Weizmann Institute, Israel [1]. In their experiment, an array of ridge waveguides were etched onto an optical substrate. In the experiments, at low power, linear behavior is observed; see Figure 1-top. The light is found to spread among nearly all the waveguides hence experiencing discrete diffraction. However, at sufficiently high power, the beam self-traps to form a localized state (a soliton) in the center waveguides see Figure 1 -bottom.
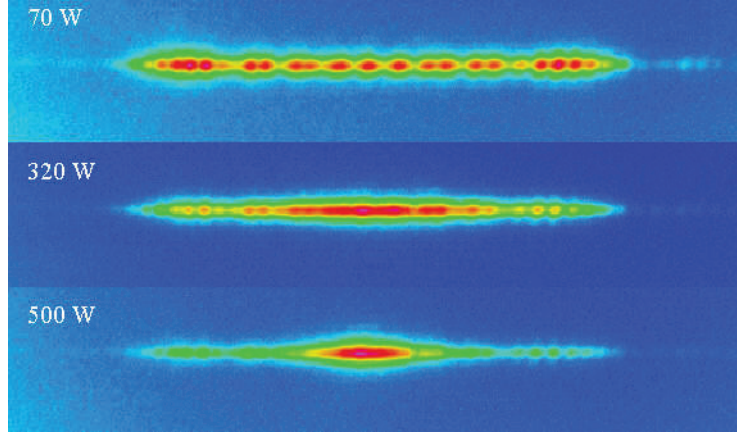

Figure 1: Images of the output for different powers. Top figure corresponds to low peak power; linear features are demonstrated. Middle inset is taken at moderate peak power where it is seen that the pattern is shrinking. Bottom figure corresponds to high peak power At sufficiently high power, a localized structure, i.e. a discrete soliton, is formed.

Subsequently many theoretical studies of discrete solitons in waveguide arrays which found switching, steering and other interesting properties.

This motivated many interesting properties of nonlinear lattices and discrete solitons including switching, steering and other interesting properties. Discrete systems exhibit phenmomena that are absent in continuous media such as the possibility of producing *anomalous* diffraction [5]. Hence, in discrete systems self-focusing and defocusing processes can be achieved in the same medium (structure) and wavelength. This also leads to the possibility of observing discrete dark solitons in self-focusing Kerr media [6]. The experimental observations of discrete solitons [1] and diffraction

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management [5] also motivated interest in discrete solitons in nonlinear lattices.

In [7] we considered the possibility of moving discrete solitons; we found that as discretization effects become important the solitons slowed down. This work motivated our studies of more complicated situations in which laser beams can propagate in a discrete medium with alternating diffraction. A natural question which arises is: What happens if we launch a high power laser beam into a diffraction managed waveguide array [8]? We found modes which "breathe" periodically upon propagation. During propagation the peak amplitude, FWHM, and phase of the beam evolve. For example, during each map period, the FWHM of the beam oscillates between a nonzero minimum and a maximum value while maintaining conservation of power.

By designing the diffraction properties of a linear waveguide array, additional new phenomena are found to occur. In [9] we showed that by appropriately tailoring the diffraction properties of a waveguide array, the interaction between discrete solitons can be altered to achieve better tunability and control over the collision outcome. For example, by colliding vector discrete solitons, remarkably the interaction picture involves beam shaping, fusion and fission. Interestingly it was also shown that discrete solitons in two-dimensional networks of nonlinear waveguides can be used to realize intelligent functional operations cf. [10].

Two dimensional periodic nonlinear photonic lattices

In many photonics applications a nonlinear Schrödinger equation with an external potential, or the lattice NLS equation, is the governing equation. It can be derived from Maxwell's equations, and in dimensionless form it is given by

$$iu_z + \Delta u - V(\mathbf{r})u + |u|^2 u = 0 \ . \tag{1}$$

Here $u(\mathbf{r}, z)$ corresponds to the slowly-varying complex amplitude of the electric field in the $\mathbf{r} = (x, y)$ plane propagating along the z direction, Δu corresponds to diffraction, $V(\mathbf{r})$ is an optical potential representing the linear index of refraction and the cubic nonlinear term originates with the nonlinear-Kerr change of the refractive index often referred to as the $\chi^{(3)}$ nonlinear index.

The potential is assumed uniform along the propagation direction z, although non-uniform

potentials in z could also be considered in a manner analogous to diffraction or dispersion management, and more recently has been employed in a novel way in honeycomb lattices [11].

Most studies consider $V(\mathbf{r})$ to be periodic in $\mathbf{r} = (x, y)$, though some work on lattice defects have been carried out cf. [12]. When the lattice, $V(x, y)$ is given by

$$V(x, y) = \frac{V_0}{N^2} |\sum_{0}^{N-1} e^{i(k_x x + k_y y)}|^2 \tag{2}$$

where $k_x = k \cos(2\pi n/N), k_y = k \sin(2\pi n/N), k$ are constant, it is periodic with $N = 4$; this is sometimes called a square lattice. Aperiodic lattices or quasi-crystals arise when $N = 5, 7$.

In [13] solitons were observed in "irregular" quasi-crystal lattices (e.g. N=5) and in [14] a detailed computational study carried out for this potential and for other potentials with defects. In [15] we studied the stability properties and the boundaries of the bandgaps associated with the above potential for the lowest band of the dispersion surface; and in [16] we computationally investigated the existence and stability of vortex solitons.

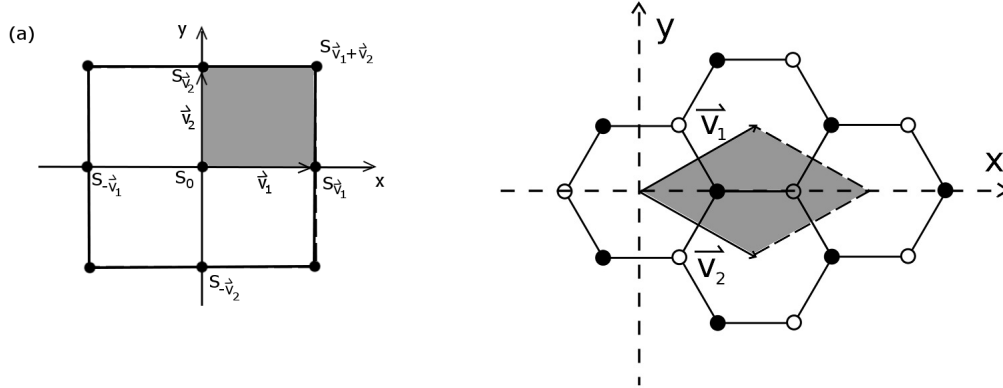Next we consider certain two-dimensional periodic optical external potentials:

Figure 2: Left: simple square lattice; right: non-simple honeycomb (HC) lattice

$$V(\mathbf{r} + m\mathbf{v_1} + n\mathbf{v_2}) = V(\mathbf{r})$$

with $m, n \in \mathbb{Z}$, separated into two categories: simple and non-simple honeycomb (HC) lattices. In a simple lattice all sites can be constructed from one site, whereas non-simple HC lattices are

constructed from more than one site; Figure 2 illustrates a simple square lattice and a non-simple honeycomb lattice.

In the case of a simple square lattice, the analog experiment described above in the context of one dimensional waveguides, was explored. At low power an input beam diffracts whereas at high power a 2d lattice soliton is observed [17]; see: Figure 3– experiment, and Figure 4 where computational results were obtained by solving equation (1). Subsequently there have been many studies which have investigated a variety of localized soliton solutions in two-dimensional simple lattices cf. [18, 19, 20, 21, 22]
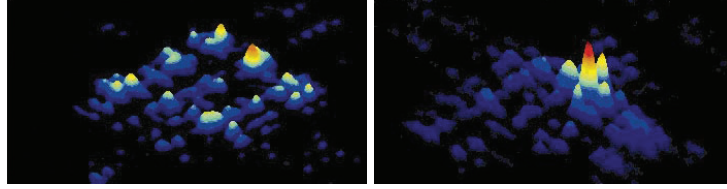
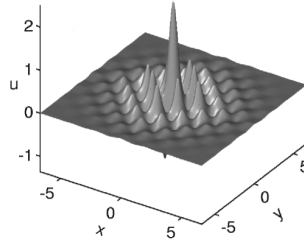Figure 3: Experiment: Left: Low Power; Right: High Power

Figure 4: Typical 2d lattice soliton: Computational

In what follows next, we describe some of our recent work involving two dimensional optical lattices.

The underlying linear problem, is obtained from equation (1) by assuming $u(\mathbf{r}, z) = \varphi(\mathbf{r})e^{-i\mu z}$ with $|\varphi| \ll 1$,; this yields the linear Schrödinger equation with periodic coefficients:

$$(\nabla^2 - V(\mathbf{r}))\varphi = -\mu\varphi \tag{3}$$

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where $V(\mathbf{r})$ is a 2-d periodic potential with lattice vectors: $\mathbf{v}_1, \mathbf{v}_2$ on the lattice $\mathbb{P} = \{m\mathbf{v}_1 + n\mathbf{v}_2 : m, n \in \mathbb{Z}\}$.

Bloch theory implies that solutions φ of equation (3), called Bloch modes, have the form $\varphi = \varphi(\mathbf{r}; \mathbf{k}) = e^{i\mathbf{k} \cdot \mathbf{r}} U(\mathbf{r}; \mathbf{k})$ where $\mathbf{k}$ is real, $U(\mathbf{r}; \mathbf{k})$ is periodic in $\mathbf{r}$ with the same periodicity as $V(\mathbf{r})$ and $\varphi(\cdot, \mathbf{k}), \mu = \mu(\mathbf{k})$ are periodic in $\mathbf{k}$ with 'dual' lattice vectors: $\mathbf{k}_1, \mathbf{k}_2$ where $\mathbf{v}_m \cdot \mathbf{k}_n = 2\pi \delta_{mn}$; $\mu = \mu(\mathbf{k})$ is called the dispersion relation.

Due to periodicity in $\mathbf{k}$ we can represent $\varphi(\mathbf{r}; \mathbf{k})$ as $\varphi(\mathbf{r}; \mathbf{k}) = \sum_{\mathbf{v}} \phi_{\mathbf{v}}(\mathbf{r}) e^{i\mathbf{k} \cdot \mathbf{v}}$ where the $\sum_{\mathbf{v}}$ is over all values in $\mathbb{P}$; $\phi_{\mathbf{v}}(\mathbf{r})$ is termed a 'Wannier function'. Using the physical space periodicity it follows that $\phi_{\mathbf{v}}(\mathbf{r}) = \phi_{\mathbf{0}}(\mathbf{r} - \mathbf{v})$.

In general one cannot find explicit analytical solutions for these Wannier functions. But when the potential is large, i.e. $|V| \gg 1$, which is termed the tight binding approximation (TBA) and is well studied in the literature, one can approximate the Wannier functions. For simple lattices Wannier functions are approximated by "orbitals" via:

$$\left(\nabla^2 - V_0(\mathbf{r})\right) \phi(\mathbf{r}) = -E\phi(\mathbf{r})$$

where $V_0(\mathbf{r})$ is the potential in the fundamental cell, constant outside; $\phi(\mathbf{r}), E$ are the corresponding eigenfunction (orbital) and eigenvalue. The approximation involves solving for $\phi(\mathbf{r}), E$ in the limit $|V| \gg 1$; it is often useful to consider $V_0(\mathbf{r}) \approx D_0(a^2(x^2 + c^2 y^2))$ locally near the minima; in this case at the origin.

In [23, 24] we developed the necessary mathematical foundations to study nonlinear waves in simple and non-simple HC periodic lattices.

Simple lattices

For simple lattices with one dispersion branch we can find discrete evolution equations for the envelope associated with the lattice NLS equation (1) by looking for solutions of the form

$$u \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(Z)\phi(\mathbf{r} - \mathbf{v})e^{i\mathbf{k} \cdot \mathbf{v}}$$

where $Z = \varepsilon z$, $|\varepsilon| \ll 1$ and $a_{\mathbf{v}}(Z)$ represents the slowly varying Bloch wave envelope at the site $S_{\mathbf{v}}$. Substituting this approximation into the lattice NLS equation (1), multiplying by $\phi(\mathbf{r} - \mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}$, integrating over all space after some manipulation yields a discrete NLS equation at general values of $\mathbf{k}$

$$i\varepsilon\frac{da_{\mathbf{p}}}{dZ} - \sum_{<\mathbf{v}>}a_{\mathbf{p}+\mathbf{v}}C_{\mathbf{v}}e^{i\mathbf{k}\cdot\mathbf{v}} + \sigma g|a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0$$

where ϕ is an orbital (see above), $g = \int[\phi(\mathbf{r})]^4 d\mathbf{r}$ and $<\mathbf{v}>$ means we take nearest neighbors to point $\mathbf{p}$. This is a generalization of the well-known one dimensional lattice problem [4], but extended to two dimensional square and triangular lattices [23]. These results can be extended to cases where there are more than one dispersion relation band.

As a further limit, when the envelope $a_{\mathbf{v}}$ varies slowly over $\mathbf{v}$ with a scale $\mathbf{R} = \nu\mathbf{r}$, $|\nu| \ll 1$; then

$$\psi \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(Z)\phi(\mathbf{r} - \mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} \approx \sum_{\mathbf{v}} a(\mathbf{R}, Z)\phi(\mathbf{r} - \mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}}$$

The continuous limit yields a continuous NLS equation for each $\mathbf{k}$ point in the Brillouin zone

$$i\varepsilon\frac{\partial a}{\partial Z} + i\nu\bar{\nabla}\mu \cdot \tilde{\nabla}a + \frac{\nu^2}{2}\sum_{m,n=1}^{2}\bar{\partial}_{m,n}\mu\tilde{\partial}_{m,n}a + \sigma g|a|^2 a = 0.$$

where $\tilde{\partial}_m = \frac{\partial}{\partial \mathbf{R}_m}, \bar{\partial}_m = \frac{\partial}{\partial k_m}$ and $\bar{\nabla}\mu$ plays the role of the group velocity. For different locations in a Brillouin zone (periodic cell in $\mathbf{k}$ space) the spatial operator in the above NLS equation can be elliptic, hyperbolic or parabolic. We remark that further reduction is possible by going into a moving frame with the group velocity and introducing a slower evolution scale $\tilde{Z} = \nu Z$; maximal balance is obtained when $\varepsilon = O(\nu) = O(\sigma g)$.

Non-simple honeycomb lattices

Next we consider background honeycomb (HC) lattices such as that depicted in Figure 2–right; this was considered in photonic lattices by Segev's group cf. [25, 26]. A typical intensity plot of a honeycomb lattice is given in Figure 5 where the local minima are in the blue regions, which in turn form an hexagonal lattice.
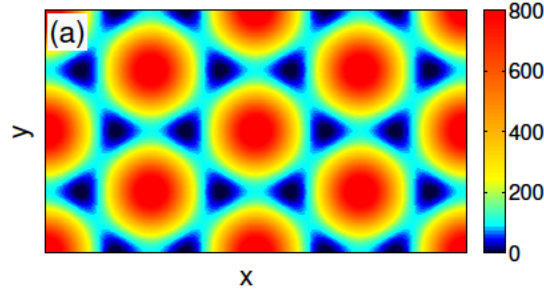
Figure 5: Typical physical space intensity plot of a honeycomb lattice

Figure 6: First two bands of the dispersion relation $\mu(\mathbf{k})$ associated with a typical honeycomb lattice; the bands touch at Dirac points

Similarly a typical HC dispersion relation $\mu(\mathbf{k})$ for the lowest two bands is given in Figure 6. It is seen that the bands touch (cf. [27, 28]) at special points, called Dirac points which in turn form the edges of a HC lattice.

In the carbon based material graphene Dirac points play an important role in terms of the novel phenomena found such as extreme conductance and strength. Similarly, in optical applications new phenomena are found. In optics this research direction is sometimes referred to as photonic graphene or optical graphene. This is an artificial form of its carbon based analogue; it allows one to investigate phenomena which would be very difficult to do in its original carbon form.

In our research [29, 30] we have found novel discrete equations associated with HC lattices via the tight binding approximation. The solution of the lattice NLS equation is assumed to be of the

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form

$$u \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(Z)\phi_A(\mathbf{r} - \mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} + \sum_{\mathbf{v}} b_{\mathbf{v}}(Z)\phi_B(\mathbf{r} - \mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}}$$

where $Z = \varepsilon z$, $|\varepsilon| << 1$ and sum $\mathbf{v}$ takes all values $m, n \in \mathbb{Z}$ associated with A, B (minima in each cell) sites respectively and the orbitals ϕ_A, ϕ_B satisfy

$$\left(\nabla^2 - V_j(\mathbf{r})\right)\phi_j(\mathbf{r}) = -E_j\phi_j(\mathbf{r}); \quad j = A, B$$

Substituting u into the lattice NLS equation (1), multiplying by $\phi_j(\mathbf{r} - \mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}$; $\quad j = A, B$ and integrating we find after some manipulation and normalization (with maximal balance: slow evolution, dispersion, nonlinearity) the following Discrete HC system, **at a general location** k:

$$i\frac{da_{\mathbf{p}}}{dZ} + \mathcal{L}^-b_{\mathbf{p}} + \tilde{\sigma}|a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0 \tag{4}$$

$$i\frac{db_{\mathbf{p}}}{dZ} + \mathcal{L}^+a_{\mathbf{p}} + \tilde{\sigma}|b_{\mathbf{p}}|^2 b_{\mathbf{p}} = 0 \tag{5}$$

$$\mathcal{L}^-b_{\mathbf{p}} = b_{\mathbf{p}} + \rho b_{\mathbf{p}-\mathbf{v}_1}e^{-i\mathbf{k}\cdot\mathbf{v}_1} + \rho b_{\mathbf{p}-\mathbf{v}_2}e^{-i\mathbf{k}\cdot\mathbf{v}_2}$$

$$\mathcal{L}^+a_{\mathbf{p}} = a_{\mathbf{p}} + \rho a_{\mathbf{p}+\mathbf{v}_1}e^{i\mathbf{k}\cdot\mathbf{v}_1} + \rho a_{\mathbf{p}+\mathbf{v}_2}e^{i\mathbf{k}\cdot\mathbf{v}_2}$$

where $\tilde{\sigma}$ depends on the original coefficient of nonlinearity and integrals over the orbitals, $Z = \epsilon z$, and ρ is deformation parameter which depends on given HC lattice parameters; when $\rho = 1$ we have perfect hexagonal HC lattice. The effect of ρ in physical space is to squeeze the bottom/top hexagonal corners of the HC lattice when $\rho < 1$ and push them apart when $\rho > 1$; when $\rho < 1/2$ a gap opens at the Dirac points.

The above system is a discrete evolution system which governs the evolution of Bloch wave envelopes at a general location k. For different values of k, we find that the evolution of the envelopes can be very different. We can consider a further limit like we did for simple lattices; i.e. $a_{\mathbf{p}+\mathbf{v}} \sim (1 + \nu\mathbf{v}\cdot\nabla + \cdots)a$; namely the continuous limit where the envelopes can be written in the

form $a(\mathbf{R}, Z)$ and $b(\mathbf{R}, Z)$ where $\mathbf{R} = \nu\mathbf{r}$ represents the spatial coordinate of the slowly varying envelope. If we assume the wave is propagating with $\mathbf{k}$ near a Dirac point we find the following continuous system of equations which is quite different from an NLS type equation

$$
\begin{aligned}
i\frac{\partial a}{\partial z} + \tilde{\partial}_- b + \tilde{\sigma}_\nu |a|^2 a &= 0 \\
i\frac{\partial b}{\partial z} - \tilde{\partial}_+ a + \tilde{\sigma}_\nu |b|^2 b &= 0
\end{aligned}
\tag{6}
$$

where $\tilde{\partial}_\pm = \partial_x \pm i\zeta\partial_y$, $\tilde{\sigma}_\nu = \frac{\tilde{\sigma}}{\nu}$ and $\zeta = \sqrt{\frac{4\rho^2-1}{3}}$. If $\tilde{\sigma}_\nu = 0$, find a linear system which is the familier 2 dimensional wave equation; otherwise this is termed a NL Dirac system.

In Figure 7 is a typical comparison between the governing lattice NLS equation and the above reduced system with $\rho = 1$. In Figure 7 top panel a-c: the intensity of the solution is plotted while in the lower panel d-f: only the a component of the continuous Dirac system (6) is given– the b component looks similar to a. The initial condition is a unit gaussian for a and zero for the b component. We see bright rings emanating from the center; this is termed conical diffraction. Interestingly, similar conical structure occurs in the linear system–i.e. when $\sigma = 0$. We note that when the initial condition for both a, b are unit gaussians we observe conical diffraction with a 'notch' [29, 30].

The main difference between the linear and nonlinear systems is the phase structure which becomes increasingly complex in the nonlinear case. Conical diffraction was observed experimentally [25]. It is interesting to note that remarkable complex phase structures were also recently observed [31].

In [32] direct computation on the lattice nonlinear Schrödinger equation (1) showed that when the nonlinear coefficient was significantly increased one obtains triangular diffraction as opposed to conical diffraction. Using our discrete system we showed [33] that triangular diffraction was present. Figure 8 describes the situation. We see that for small nonlinear coefficient the resulting pattern is nearly conical. But as the nonlinear coefficient is increased to unity a clear triangular structure emerges. When the nonlinear coefficient takes the opposite sign the triangular diffraction
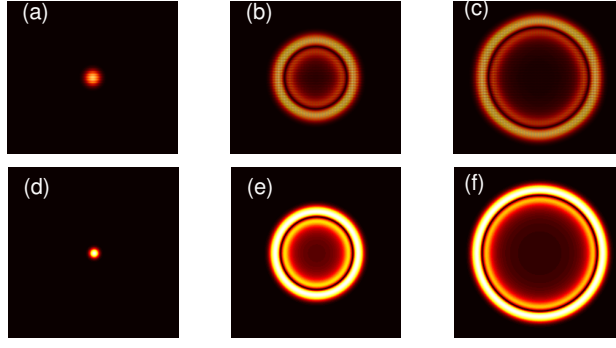
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Figure 7: An initial 'spot' evolves into bright rings; the top panel is from direct integration of the lattice NLS equation (1). The bottom panel is the a component of the above continuous Dirac system (6) with $\rho = 1$.

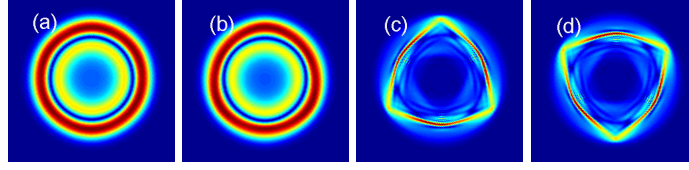changes from a triangle pointing upward to one pointing downward.

Figure 8: Effective NL coefficient in the discrete system with $\rho = 1, \tilde{\sigma} = $ (a) 0 (b) 0.1 (c) 1 (d) -1

We find that when we consider the continuous limit one needs to keep higher order terms in order to see the triangular structure. The relevant higher order continuous equations with $\rho = 1$ are given [33] by

$$i\partial_Z a + (\partial_- - \frac{\nu}{2}\Delta_+)b + \tilde{\sigma}_\nu|a|^2a = 0$$
$$i\partial_Z b + (\partial_+ - \frac{\nu}{2}\Delta_-)a + \tilde{\sigma}_\nu|b|^2b = 0$$

where $\partial_\pm = \pm\partial_X + i\partial_Y$, $\Delta_\pm = \partial_{XX} + \partial_{YY}/3 \pm 2i\partial_X\partial_Y$ and the corresponding figures demonstrating triangular diffraction are given below in Fig. 9. If the higher order terms $\Delta_\pm$ are not included in the above equations the resulting diffraction patterns are conical, not triangular.

Interestingly, when we consider initial conditions for the discrete system which are on the scale of the discretization, then in the linear discrete problem with $\tilde{\sigma} = 0$ we see spots at the hexagonal
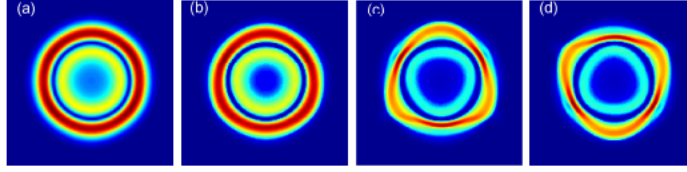
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Figure 9: The nonlinear coefficient is $\tilde{\sigma}_\nu =$ (a) 0 (b) 1 (c) 5 (d) -5
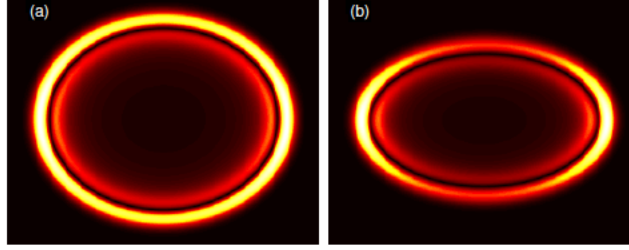

Figure 10: Circular rings become elliptic when honeycomb lattices are deformed. (a) $\rho = 0.8$, (b) $\rho = 0.6$.

corners, whereas in the nonlinear case the spots are on triangular corners with one corner spot having larger intensity, upward spot being of larger intensity for one sign of nonlinearity and visa-versa for the other sign of nonlinearity [33].

When the deformation parameter $\rho \neq 1$ then equation (6) admits an elliptical diffraction pattern which can be seen below in Figure 10. If we denote $\beta = 2\rho - 1$, then for $\beta = O(1)$ and positive we have elliptical diffraction, as mentioned above. When $\beta = O(1)$ and negative, a gap opens up in the spectrum and we find NLS type equations which can contain solitons (gap solitons). When $\rho \sim 1/2$ and $|\beta| \ll 1$ we find 'straight line' diffraction. In fact we find a class of reduced equations in a regime where $\beta = O(\nu^2)$ [34, 35]. One example of a member of this class of equations is the following

$$\partial_\theta \left(\partial_Z F - \sigma_\nu i |F|^2 F \right) + \partial_y^2 F = 0 \quad \text{NLSKZ}$$

where θ represents a right or left going characteristic variable obtained from the leading order one dimensional wave equation, obtained when $\rho = 1/2$. We refer to this equation as the NL-SKZ equation because of the analogy with the so-called classical KZ equation after Khokhlov &

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Zabolotskaya, Sov. Phys. -Acoust., 1969 given by

$$\partial_\theta(\partial_t u + u\partial_\theta u) + \gamma\partial_y^2 u = 0 \quad \text{KZ}$$

From a initial conditions for a, b of guassian type we find that there are two small parabolic structures emanating from the center. The asymptotic theory is found to provide a good approximation to the underlying complex situation. More details can be found in our papers [34, 35].

Recently it has been shown [11] that semi-infinite HC lattices with a periodic helical structure written along the propagating direction can support linear traveling edge modes. The written helical HC structure has the effect of introducing a pseudo-magnetic field. In [11] it is found that these linear edge modes can move along and follow the boundary even when there are corners. In preliminary work we have developed an asymptotic theory, assuming a relatively fast helical change, to describe this situation. We first studied the case without a pseudo-magnetic field present; i.e. the existence of edge modes in system (4). In this case we found a class of stationary edge corresponding to so-called zig-zag boundary conditions cf. [36] and references therein. In [37] we developed an asymptotic theory that describes and extends the results of [11] to the case where there is a pseudo-magnetic field. The advantage of the asymptotic theory is that all formulae are explicit which allows us to readily generalize to other cases of physical interest.

In addition to studying the dynamics and propagation of edge wave dynamics in honeycomb lattices we also analyzed the propagation of multidimensional nonlinear surface plasmon waves in dielectric/metal interfaces [38]. Knowledge of both areas is important in order for us to deeply understand the nature of wave dynamics along edges and surfaces.

Using the HC lattice theory with a pseudo-field we were able to examine situations with more general pseudo-magnetic fields. We find that the linear edge modes are remarkably robust in a certain regime of parameter space; in other regimes they disperse. We have also used the theory to generalize to nonlinear lattices and have found that the envelope function associated with the edge modes satisfy the classical nonlinear Schrödinger (NLS) equation. We will continue studying this

problem and will carefully investigate whether the nonlinear soliton states can propagate without dispersion along boundaries even when there are corners present.

The robust behavior of the propagating edge modes appears to be correlated with underlying topology associated with the Bruillouin zone [11, 39]. Propagating edge modes persist over long distances despite the fact that we are in a gap regimes of the edge mode spectrum. This situation/effect is sometime referred to as a 'Floquet topological insulator' in the physics literature. Topological insulators are insulators in the bulk but admit boundary conducting sates whose energies lie in a bulk energy gap. The term Floquet derives from the fact that the pseudo-field in periodic in the longitudinal direction. Preliminary calculations indicate that the robustness of the Floquet topological insulator in the linear case can also lead to similar results for the nonlinear regime. We are continuing to study this in detail.

PT-Symmetric systems

Another area of interest in modern optics is so called $\mathcal{PT}$ systems. These systems are made by introducing gain and loss in just such a way that the equations describing the propagation of fields which are invariant under the combined action of spatial inversion, $\mathcal{P}$, and time reversal, $\mathcal{T}$. $\mathcal{PT}$-symmetric optical systems represent a new class of optical metamaterials allowing for greater control of light. In particular, $\mathcal{PT}$-symmetric optical systems, phenomena such as novel beam diffraction patterns and unidirectional invisibility have been observed. All of this phenomena depends on the fundamental question of whether spectra is real or if a 'phase transition' [40] has caused spectra with non-zero imaginary part to appear. Establishing whether phase-transitions do or do not occur has been studied in many contexts cf. [41]. In [42], starting from a linear Schrödinger equation of the type discussed above, see eq. (1) with a PT symmetric lattice potential, we develop a necessary and sufficient condition which establishes when phase transitions can or can not occur in $\mathcal{PT}$ symmetric honeycomb lattices for small $\mathcal{PT}$ perturbations. We we show that honeycomb potentials with added symmetry allow us to find $\mathcal{PT}$ perturbations which satisfy this condition. Via numerical experiments, it is seen that $\mathcal{PT}$-symmetric lattices satisfying the analytic

condition do not exhibit phase transitions for a range of parameter regimes, sometimes even for relatively large $\mathcal{PT}$ perturbations. This goes beyond the standard theory, and shows with added symmetry that carefully designed $\mathcal{PT}$ symmetric lattices can be robust against phase transitions.

In a related direction we have found that a 'simple looking' nonlocal PT-symmtric equation is integrable. The equation takes the form

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q(x,t)q^*(-x,t)q(x,t) \tag{8}$$

where $*$ denotes complex conjugation and $q(x,t)$ is a complex valued function of the real variables x and t. Eq. (8) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws; hence it is an integrable system. Using the the inverse scattering transform we can linearize the equation, corresponding to rapidly decaying initial data, and obtain solutions to Eq. (8) including pure solitons solutions [43]. Some of the important properties of the nonlocal NLS equation are contrasted with the classical NLS equation where the nonlocal nonlinear term $q^*(-x,t)$ is replaced by $q^*(x,t)$. Indeed we note that both equation (8) and the classical NLS share the symmetry that when $x \to -x$, $t \to -t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is PT symmetric which, in the case of classical optics, amounts to the invariance of the induced potential $V(x,t) = q(x,t)q^*(-x,t)$ under the combined action of parity and time reversal symmetry. Finally, wave propagation in PT symmetric coupled waveguides/photonic lattices has been experimentally observed in classical optics cf. [40].

Similarly we find the following, also 'simple looking', nonlocal discrete nonlinear Schrödinger equation is integrable [44]

$$i\frac{dq_n}{dt} = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) \pm q_n q_{-n}^* \left(q_{n+1} + q_{n-1}\right) \tag{9}$$

where $*$ denotes complex conjugation and q_n is a complex function and n is an integer. As with equation (8) this discrete equation (9) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws; hence it too is an integrable system. Corresponding to rapidly

decaying initial data one can *exactly* linearize Eq. (9) using the inverse scattering transform and obtain solutions including pure solitons. Some of the important properties of Eq. (9) are contrasted with the better known model where the nonlocal nonlinear term $q_n q_{-n}^*$ is replaced by $|q_n|^2$. Indeed as with the continuous model, the discrete equation shares the symmetry that when $n \to -n$, $t \to -t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new discrete nonlocal equation is PT symmetric which, in the case of classical optics, amounts to the invariance of the self-induced potential $V_n = q_n q_{-n}^*$ under the combined action of parity and time reversal symmetry.

Dynamics of ultra-short mode-locked laser pulses

Research developments with mode-locked lasers, such as Ti:sapphire lasers, have enabled scientists to generate regularly spaced trains of ultrashort pulses, which are separated by one cavity round-trip time. Fig. 11 below shows a schematic of a mode-locked Ti:sapphire laser and the emitted pulse train.
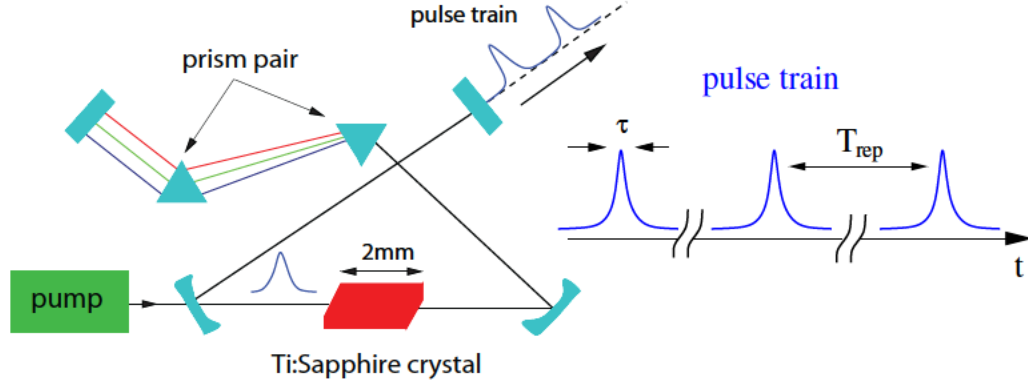

Figure 11: Ti:sapphire laser (left) and the emitted pulse train (right).

Associated with the spectrum, or the Fourier transform of the pulse train, is a frequency comb whose frequencies are separated by the laser's repetition frequency. Progress in the development of optical oscillators has been made possible by controlling these femtosecond frequency combs. Extremely stable frequency combs have been generated by Ti:s laser systems, but other types of mode-locked lasers such as Sr:Forsterite and fiber lasers are also being actively studied. It is important to have useful mathematical models of these laser systems.

In our research we have been studying a distributed dispersion-managed equation which we term the power energy saturation (PES) model. For a pulse amplitude $u(z,t)$, power $P(z,t) = |u|^2$, and energy $E(z) = \int_{-\infty}^{+\infty} |u|^2 \, dt$, propagating in the z direction, the normalized or dimensionless equation we study takes the form

$$i\frac{\partial u}{\partial z} + \frac{d(z)}{2}\frac{\partial^2 u}{\partial t^2} + n(z)|u|^2 u = \frac{ig}{1 + E/E_{sat}}u + \frac{i\tau}{1 + E/E_{sat}}u_{tt} - \frac{il}{1 + P/P_{sat}}u \qquad (10)$$

where the constant parameters g, τ, l, E_{sat}, P_{sat} are positive. The first term on the right hand side represents saturable gain, the second is nonlinear filtering ($\tau \neq 0$) and the third is saturable loss. This model generalizes the well-known master laser equation originally developed by Haus and collaborators. When the loss term is approximated in the weakly-nonlinear regime by a first order Taylor polynomial we obtain the master laser equation. Hence, the master laser equation is included in the power saturated model as a first order approximation.

In our papers we discuss in detail how this equation applies to the system described in Fig. 11. The system is dispassion managed because there is a difference between the dispersion inside the crystal and in the prisms/mirrors.

In our earlier research investigations in fiber optics [45] we derived, based on the asymptotic procedure of multiple scales, a nonlinear integro-differential equation (not given here due to space considerations) which governs the dynamics of dispersion-managed pulse propagation. This governing equation is referred to as the dispersion-managed nonlinear Schrödinger (DMNLS) equation. When there is no gain or loss in the system, for strongly dispersion-managed systems, the DMNLS equation plays the role of the "pure" NLS equation–which is the relevant averaged equation when there is either small or no dispersion-management. When gain and loss are included as in the PES equation we find a modification of the "pure" DMNLS equation [46].

With or without dispersion-management the PES equation naturally describes the locking and evolution of pulses in mode-locked lasers that are operating in the soliton regime. In our research [46, 47] we took a unit gaussian input, typical values of the parameters, and we varied the gain parameter g. When $g < g^*$ no localized solution was obtained; i.e. in this case the effect of loss is stronger than a critical gain value and the evolution of a Gaussian profile decays to the trivial solution. Conversely, when $g > g^*$, there exists a single localized solution, $u = U_0(t)\exp(i\mu z)$ where μ, called the propagation constant, is uniquely determined given the specific values of the other parameters.

A typical situation is described by the evolution of the pulse peak for different values of the gain parameter g as shown in Fig. 12; here we keep all terms constant and only change the gain parameter g. When $g = 0.1$ the pulse vanishes quickly due to excessive loss with no noticeable os-
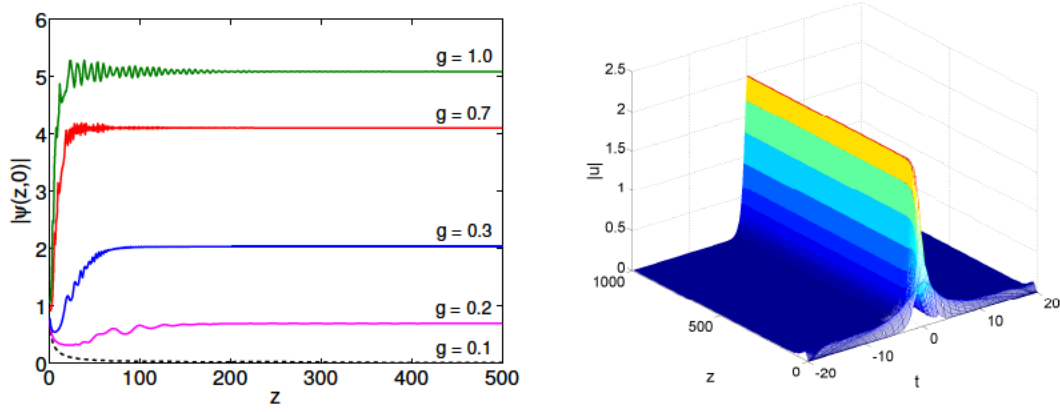

Figure 12: Evolution of the pulse peak of an arbitrary initial profile under PES with different values of gain. The damped pulse-peak evolution is shown with a dashed line. In the right hand figure, the complete evolution is given for $g = 0.3$.

cillatory behavior; the pulse simply decays, resulting in damped evolution. When $g = 0.2, 0.3$, due to the loss in the system the pulse initially undergoes a relative to its amplitude a modest decrease. However, it recovers and evolves into stable solution. Interestingly, e.g. when $g = 0.7, 1$, and the perturbations are not small, a stable evolution is nevertheless again obtained, although some-what different from the case above. Now with excessive gain in the system, the pulse amplitude increases develops oscillations but a steady state is rapidly reached.

Generally speaking in the PES model, the mode-locking effect is present for $g \geq g^*$, a critical gain value. Without enough gain i.e. $g < g^*$, pulses dissipate to the trivial zero state. Furthermore, we do not find complex radiation states or states whose amplitudes grow without bound for pa-rameter regimes we studied. In terms of solutions, Eq. (10) admits soliton states for all values of $g \geq g^* > l$ (recall, here $l = 0.1$). This was also shown to be the case when we employed analytical methods (soliton perturbation theory).

It is useful to note that only for a narrow range of parameters does the master laser equation

(when the loss term is taken to be the first two terms of the Taylor expansion of the last, power saturated term, in the PES equation) have stable soliton solutions or mode-locking evolution. In general the solitons are found to be unstable; either dispersing to radiation or evolving into nonlocalized quasi-periodic states. For different parameters, the amplitude can also grow rapidly under evolution. Thus, the basic master laser equation captures some qualitative aspects of pulse propagation in a laser cavity; however, since there is only a small range of the parameter space for which stable mode-locked soliton pulses exist, it does not reflect the wide ranges of operating conditions where mode-locking occurs.

Interestingly, as the gain becomes stronger additional soliton states are possible and 2, 3, 4 or more coupled pulses are found to be supported. This means that strings of soliton states can be obtained [48].

Power saturation models also arise in other problems in nonlinear optics and are central in the underlying theory. For example, power saturation models are important in the study of the dynamics of localized lattice modes (solitons, vortices, etc) propagating in photorefractive nonlinear crystals. If the nonlinear term in these equations was simply a cubic nonlinearity, without saturation, two dimensional fundamental lattice solitons would be vulnerable to blow up singularity formation, which is not observed. Thus saturable terms are crucial in these problems.

In recent work we analyzed how the mode-locking mechanism responds to higher order perturbative terms including third order and nonlinear dispersion and Raman gain; we considered the following perturbation of the PES equation (10) with constant dispersion and nonlinearity

$$
i\frac{\partial u}{\partial z} + \frac{d_0}{2}\frac{\partial^2 u}{\partial t^2} + |u|^2 u = \frac{igu}{1 + E/E_{\text{sat}}} + \frac{i\tau}{1 + E/E_{\text{sat}}}\frac{\partial^2 u}{\partial t^2} - \frac{ilu}{1 + P/P_{\text{sat}}}
$$
$$
+ i\beta\frac{\partial^3 u}{\partial t^3} - i\gamma\frac{\partial(|u|^2 u)}{\partial t} + Ru\frac{\partial(|u|^2)}{\partial t} \tag{11}
$$

where β represents third order dispersion, γ nonlinear dispersion or self-steepening and R the Raman gain. When $\beta = \gamma = R = 0$ this equation is the PES model. In [49] mode-lockiing was found to be maintained even with these higher order terms terms; mode-locking occurs for

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both the anomalous and normal regimes. In the anomalous regime, these perturbations are found to affect the speed but not the structure of the locked pulses. The pulses behave like solitons of a classical nonlinear Schrödinger equation and as such a soliton perturbation theory verifies the numerical observations. In the normal regime, the effect of the perturbations is small, in line with recent experimental observations [50].

Dark Solitons and mode locked lasers

Dark solitons or envelope solitons having the form of density dips with a phase jump across their density minimum, are fundamental nonlinear excitations which arise for example, in the defocusing nonlinear Schrödinger (NLS) equation. They are termed *black* if the density minimum is zero or *grey* otherwise. The discovery of these structures, which dates back to early 1970s, was followed by intensive study both in theory and in experiment: in fact, the emergence of dark solitons on a modulationally stable background is a fundamental phenomenon arising in diverse physical settings. Dark solitons have been observed and studied in numerous physical contexts including: discrete mechanical systems electrical lattices, magnetic films, plasmas, fluids, atomic Bose-Einstein condensates as well as nonlinear optics.

In nonlinear optics, dark solitons have some advantages as compared to their bright counterparts, which are solutions of the focusing NLS mode. Dark solitons are more robust against Gordon-Haus jitter, background noise and higher order dispersion.

Recently, there has been an interest in "dark pulse lasers", namely laser systems emitting trains of dark solitons as envelopes of a continuous wave (cw) emitted by the laser; various experimental results have been reported utilizing fiber ring lasers, quantum dot diode lasers and dual Brillouin fiber lasers [51, 52]. These works, apart from introducing a method for a systematic and controllable generation of dark solitons, can potentially lead to other important applications related, e.g. with optical frequency combs, optical atomic clocks, fiber optics etc. An important aspect in these studies is the ability of the laser system to induce a fixed phase relationship between the modes of the laser's resonant cavity, i.e. to *mode-lock*; in such a case, interference between the laser modes in the normal dispersion regime causes the formation of a sequence of dark pulses on top of the stable cw background emitted by the laser.

We have studied dark solitons subject to general perturbations

$$iu_z - \frac{1}{2}u_{tt} + |u|^2 u = F[u] \tag{12}$$

where $|F| \ll 1$. We have developed a general perturbation theory for dark solitons [53]. One of

our results is that, in general, perturbations of dark solitons induce a small but wide shelf. We can calculate the perturbation to the soliton parameters as well as the shelf.

As a prototypical situation involving dark solitons, we considered the perturbation theory within the framework of the PES equation (10), in the normal regime [54]. We found that general initial conditions evolve/mode-lock into dark solitons under appropriate gain requirements. The resulting pulses are essentially dark solitons of the unperturbed nonlinear Schrödinger equation. In a ring like laser system the PES model also develops a rather pronounced shelf. In Figure (13) below, a typical situation is depicted for a ring laser system described by the PES equation. The shelf is seen to be prominent in the PES equation. No shelf exists in the unperturbed NLS equation. In the inset of the figure the phase change across the dark soliton is given for the PES model vs. the unperturbed NLS equation. In practice the shelf might interact with background noise creating a complex background upon which the dark soliton resides. Nevertheless dark solitons are extremely stable even with such background disturbances.
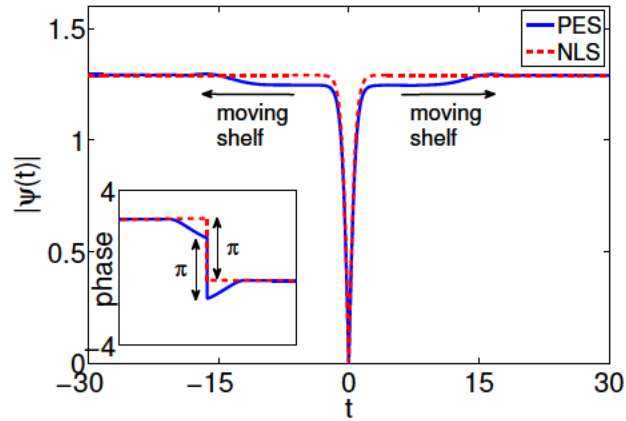
Figure 13: The development of a shelf in the solution to the PES equation in a ring laser configuration (solid line) as compared to the solution of the NLS equation (dashed line). The shelf is evident in the amplitude. The inset shows how the phase changes across the soliton.

Dispersive Shock Waves

Shock waves in compressible fluids is a classically important field in applied mathematics and physics, whose origins date back to the work of Riemann. Such shock waves, which we refer to as classical or viscous shock waves (VSWs), are characterized by a localized steep gradient in fluid properties across the shock front. Without viscosity one has a mathematical discontinuity; when viscosity is added to the equations, the discontinuity is "regularized" and the solution is smooth. An equation that models classical shock wave phenomena is the Burgers equation

$$u_t + uu_x = \nu u_{xx} \tag{13}$$

If $\nu = 0$, we have the inviscid Burgers equation which admits wave breaking. When the underlying characteristics cross a discontinuous solution, i.e. a shock wave, is introduced which satisfies the Rankine-Hugoniot jump conditions which, in turn, determines the shock speed. Analysis of Burgers equation shows that there is a smooth solution given by

$$u = \frac{1}{2} - \frac{1}{2}\tanh\{\frac{1}{4\nu}(x - \frac{1}{2}t)\}$$

which tends to the shock solution as $\nu \to 0$. Thus the mathematical discontinuity is regularized when viscosity ν is introduced. A typical regularized shock wave can be seen in Fig. 14 below in red.

Another type of shock wave is a so-called dispersive shock wave (DSW). Early observations of DSWs were ion-acoustic waves in plasma physics; indeed the Korteweg-deVries (KdV) equation describes ion-acoustic waves in plasmas. Subsequently, Gurevich and Pitaevskii [55] studied the small dispersion limit of the KdV equation. They obtained an analytical representation of a DSW. As opposed to a localized shock as in the viscous problem, the description of a DSW is one with a sharp front with an expanding, rapidly oscillating rear tail. The Korteweg-deVries (KdV) equation with small dispersion is given by

$$u_t + uu_x = \epsilon^2 u_{xxx} \tag{14}$$

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where $|\epsilon| \ll 1$ regularizes the discontinuity that otherwise would be present. The mathematical technique used to analyze DSWs relies on wave averaging, often referred to as Whitham theory. Whitham theory is used to construct equations for the parameters associated with slowly varying wavetrains; it provides an analytical basis for DSW dynamics. For KdV the Whitham equations can be transformed into Riemann-invariant form, which can be analyzed in detail. The picture and details are quite different from viscous shock waves.

A viscous shock wave which occurs in Burgers equation is depicted below in Fig. 14–in red; a typical DSW associated with the KdV equation is illustrated in blue. For the KdV DSW one finds that there are two speeds associated with a DSW: one is the speed associated with the frontal wave which is a soliton (located at x_s in the figure), and the other speed corresponds to the group velocity of near linear trailing waves on the rear end (depicted by x_l in the figure) of the DSW. This is very different from the classical or viscous shock wave (located at x_c in the figure) associated with the Burgers equation. Interestingly, the structure of the KdV DSW is strikingly similar to the original plasma observations.
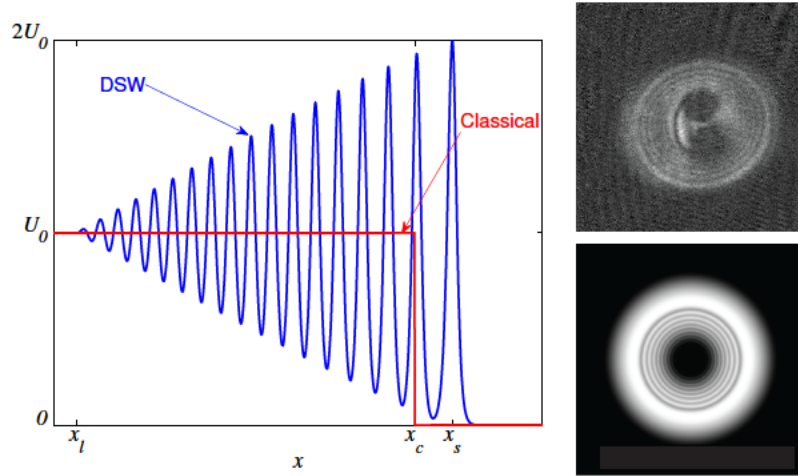

Figure 14: Left figure: typical DSW satisfying the KdV eq. (14) and a classical shock wave satisfying Burgers eq. (13); right figure: typical blast wave–experiment; numerical simulation is given below the experiment.

Recent experiments in Bose-Einstein condensates (BEC) and nonlinear optics have enhanced

interest in DSWs. The BEC experiments, originally performed in the Physics Department at the University of Colorado, motivated our studies [56]. Recent experiments in nonlinear optics carried out in the laboratory of J. Fleischer at Princeton University have also observed similar blast waves and other interesting DSW phenomena [57]. Since the equations governing the phenomena are similar, this provides further support for our DSW theory.

While interactions of viscous shock waves are well known, the understanding of interacting DSWs is still at an early stage. Nevertheless we have made some progress towards understanding the interaction picture. We note that in recent nonlinear optics experiments [57] interacting DSWs were observed. We are investigating DSW interactions in physically interesting systems by employing both Whitham methods and asymptotic analysis applied to the solution obtained by the inverse scattering transform.

Recently we have investigated the KdV equation with step-like data. We found that the long-time-asymptotic solution of the KdV equation for general, step-like data is a single-phase DSW; this DSW is the largest possible DSW based on the boundary data. We find this asymptotic solution using the inverse scattering transform and matched asymptotic expansions and confirmed the phenomena numerically. So while multi-step data evolve to have multiphase dynamics at intermediate times, these interacting DSWs eventually merge to form a single-phase DSW at large time [58, 59, 60].

Dispersive shock waves are an interesting and developing area of research which we believe will play an increasingly important role in nonlinear optics applications and other areas of physics.

PERSONNEL SUPPORTED

- Faculty: Mark J. Ablowitz

- Post-Doctoral Associates: Y. Zhu, Y-P. Ma

- Other (please list role)   None

PUBLICATIONS

- **ACCEPTED/PUBLISHED**
    - **Conference Proceedings–refereed**

    1. Nonlinear Dynamics of Bloch wave packets in honeycomb lattices, M. J. Ablowitz and Y. Zhu, Spontaneous Symmetry Breaking, Self-Trapping, and Josephson Oscillations in Nonlinear Systems, 1-26, Springer, Heidelberg (2013)

    2. Direct observation of pseudospin-mediated vortex generation in photonic graphene, D. Song, L. Tang, Y. Zhu, M. Ablowitz, V. Paltoglou, N. K Efremidis, J. Xu, Z. Chen, OSA: CLEO: QELS Fundamental Science, FM2D. 2 (2014)

    - **Journals–refereed**

    1. Nonlinear waves in shallow honeycomb lattices, M. J. Ablowitz and Yi Zhu, *SIAM J. on Appl. Math.,***72**, (2012) 240-260

    2. Vortex and dipole solitons in complex 2D nonlinear lattices, M. J. Ablowitz, N. Antar, I. Bakirtas and B. Ilan, *Phys. Rev. A*, **. 86**, (2012) 033804

    3. On tight-binding approximations in optical lattices, M.J. Ablowitz, Mark J., C.W. Curtis, Y. Zhu, *Stud. Appl. Math.,***129**, (2012) 362-388

    4. Unified Orbital Description of the Envelope Dynamics in Two-Dimensional Simple Periodic Lattices, M.J. Ablowitz and Y. Zhu, Stud. Appl. Math, **131** (2013) 41-71

5. Integrable nonlocal nonlinear Schrödinger equation, M. J. Ablowitz and Z. H. Musslimani, *Phys. Rev. Lett.* **110** (2013) 064105

6. Interactions and asymptotics of dispersive shock waves - Korteweg-de Vries Equation, M. J. Ablowitz and D. E. Baldwin, *Phys Lett. A*, **377** (2013) 555-559

7. Dark solitons of the power-energy saturation model: application to mode-locked lasers, M. J. Ablowitz, T. P. Horikis, S. D. Nixon, and D. J. Frantzeskakis, *J. of Phys. A: Math. Theor. 46* (2013) 095201

8. Dispersive shock wave interactions and asymptotics, M. J. Ablowitz and D. E. Baldwin, *Phys. Rev. E.*, **87** (2013) 022906

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10. Nonlinear wave packets in deformed honeycomb lattices, M.J. Ablowitz and Y. Zhu, *SIAM J. Appl. Math.* **87** (2013) 19591979

11. Nonlinear Temporal-Spatial Surface Plasmon Polaritons, M. J. Ablowitz and S. T. J. Butler *Optics Commun* **330** (2014) 49-55

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13. Passive mode-locking under higher order effects, T.P. Horikis and M.J. Ablowitz, *J. Opt. Soc. Am. B* **31** (2014) 2748-2753

14. Integrable discrete PT symmetric model, M. J. Ablowitz and Z. H. Musslimani, *Phys. Rev. E* **90** (2014) 032912

– **Conferences, Seminars**

– **INTERACTIONS/TRANSITIONS**

1. Invited member: 'SQuaRE: Nonlinear wave equations and integrable systems; American Institute of Mathematics (AIM), Feb. 13-17, 2012

2. Frontiers in Applied and Computational Math, NJIT, May 18-20, 2012, Nonlinear waves from beaches to nonlinear optics, May 19, 2012

3. Nonlinear evolution equations and dynamical systems (NEEDS), Chania, Greece, July 8-15, 2012, Nonlinear waves from beaches to photonic lattices, July 8, 2012

4. Nonlinear waves and Fluids, Loughborough, UK, Sept. 12-14, 2012, Nonlinear waves from fluids to photonic lattices, Sept. 12, 2012

5. AFOSR Workshop: Nonlinear Optics, Albuquerque, NM, "Optical pulse evolution and dynamics in honeycomb photonic lattices', Sept. 19, 2012

6. Invited member: 'SQuaRE: Nonlinear wave equations and integrable systems; American Institute of Mathematics (AIM) Palo Alto Calif., Feb. 18-22, 2013

7. South African Numerical and Applied Mathematics Symposium, Stellenbosch, SA, Nonlinear waves from beaches to dispersive shock waves, April 3, 2013

8. Conference Board on Math Sci. conference on two-dimensional solitons, water waves; Edinberg, Texas, May 20-24, 2013; Two-dimensional water waves theory and ocean observations, May 20, 2013

9. BIRS Workshop on Water Waves: Banff Canada, June 30-July 5, 2013; Two-dimensional water waves theory and ocean observations., July 1, 2013

10. The 8th Symposium on Quantum Theory and Symmetries, Mexico City Aug. 5-9, 2013, Nonlinear waves from beaches to dispersive shock waves, Aug. 5, 2013

11. Deformations, Localized Edge States and Associated Pulse Dynamics in Nonlinear Optical Honeycomb Lattices, AFOSR Workshop: Nonlinear Optics, Arlington, VA

12. Electrical Engineering Dept. Colloquium, North Carolina State Univ, Jan. 17, 2014, "Nonlinear waves from beaches to photonic lattices"

13. Applied Mathematics Dept., Colloquium, Calif. Inst. of Tech., Feb 3, 2014, 'Nonlinear waves from beaches to photonic lattices"

14. Center for Nonlinear Studies, Los Alamos National Laboratory, Colloquium, March 10, 2014, 'Nonlinear waves from beaches to photonic lattices"

15. Frontiers in Applied Mathematics, New Jersey Institute of Technology, May 22-23, 2014, May 22, 2014 "Nonlinear waves, solitons and applications"

16. Nonlinear Waves in Sicily, June 8-13, 2014, June 9, 2014 "Nonlinear wave in photonic lattices"

17. SIAM Conference NL Waves and Coherent Structures, Cambridge University, UK, Aug. 11-14, 2014, Aug. 11, 2014, "Nonlinear waves from beaches to photonics"

18. Nonlinear Optics Workshop, Air Force BRIC, Arlington, VA. Oct. 2, 2014, "Photonic honeycomb lattices: localized linear and nonlinear edge states"

19. University of Ioannina, "Mathematics and International Collaborations", Oct. 22, 2014

20. University of Ioannina, "Nonlinear Waves – Still Many Surprises", Oct. 23, 214

21. University of Colorado, Colorado Springs, Colorado Nonlinear days, "Nonlinear waves – Always Alluring", Nov. 1, 2014

– **Consultative and Advisory Functions to Other Laboratories and Agencies:** None

NEW DISCOVERIES, INVENTIONS, OR PATENT DISCLOSURES: None

HONORS/AWARDS:

– Named as one of the most highly cited people in the field of Mathematics by the ISI Web of Science, 2003-present

– Named Fellow of the American Mathematical Society (AMS) 2012–

– Martin Kruskal Prize/Lecture: Aug. 11, 2014, Cambridge University, Awarded by SIAM, Activity Group on Nonlinear Waves and Coherent Structures

– Doctor Honoris Causa, Oct. 2014, University of Ioannina, Greece

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1.

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Reporting Period Start Date

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Abstract

The Principal Investigators research program in nonlinear wave propagation with emphasis in nonlinear optics is very active and covers a number of research areas. During the period 15 April, 2012 -- 14 April, 2015, fourteen papers were published in refereed journals. In addition, two refereed conference proceeding were published, and twenty one invited lectures were given. The key results and research directions are described in the associated Final Report. Full details can be found in the Principal Investigators research papers which are listed at the end of the Report. Key research investigations and areas studied include the following. A detailed theory describing the dynamics of nonlinear waves in photonic honeycomb lattices both in bulk regions and with boundaries was developed. In the bulk, conical, elliptical and nearly straight-line diffraction phenomena were described. This also led to new classes of nonlocal nonlinear wave equations. When edge effects were included both linear and nonlinear localized edge waves were constructed. In certain cases, termed photonics topological insulators, the edge waves were found to be able to travel over long distances without backscatter. The detailed theory explains why photonic topological insulators are closely

related to unidirectional propagating waves. Further the theory explains why the linear results can be extended to systems which are nonlinear. The classical nonlinear Schrodinger equation was shown to describe the envelope of the nonlinear edge state. The theory agrees with and goes beyond recent experimental observations. In addition, novel continuous and discrete PT-symmetric systems and mode-locked laser systems with both bright and dark solitons, were investigated. The long time dynamics of dispersive shock waves and their interactions were also analyzed.

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Archival Publications (published) during reporting period:

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2. Vortex and dipole solitons in complex 2D nonlinear lattices, M. J. Ablowitz, N. Antar, I. Bakirtas and B. Ilan, Phys. Rev. A, Vol. 86, (2012) 033804

3. On tight-binding approximations in optical lattices, M.J. Ablowitz, Mark J., C.W. Curtis, Y. Zhu, Stud. Appl. Math., Vol. 129, (2012) 362-388

4. Unified Orbital Description of the Envelope Dynamics in Two-Dimensional Simple Periodic Lattices, M.J. Ablowitz and Y. Zhu, Stud. Appl. Math., Vol.131, (2013) 41-71

5. Integrable nonlocal nonlinear Schrödinger equation, M. J. Ablowitz and Z. H. Musslimani, Phys. Rev. Lett. Vol. 110 (2013) 064105

6. Interactions and asymptotics of dispersive shock waves - Korteweg-de Vries Equation, M. J. Ablowitz and D. E. Baldwin, Phys Lett. A, Vol. 377 (2013) 555-559

7. Dark solitons of the power-energy saturation model: application to mode-locked lasers, M. J. Ablowitz, T. P. Horikis, S. D. Nixon, and D. J. Frantzeskakis, J. of Phys. A: Math. Theor., Vol. 46 (2013) 095201

8. Dispersive shock wave interactions and asymptotics, M.J.Ablowitz and D.E.Baldwin, Phys. Rev. E., Vol. 87 (2013) 022906

9. Localized nonlinear edge states in honeycomb lattices, M.J. Ablowitz and C.W. Curtis, Phys. Rev. A, Vol 88 (2013) 013850

10. Nonlinear wave packets in deformed honeycomb lattices, M.J. Ablowitz and Y. Zhu, SIAM J. Appl. Math., Vol. 87 (2013) 1959-1979

11. Nonlinear Temporal-Spatial Surface Plasmon Polaritons, M. J. Ablowitz and S. T. J. Butler, Optics Commun. Vol. 330 (2014) 49-55

12. Linear and nonlinear traveling edge waves in optical honeycomb lattices MJ Ablowitz, CW Curtis and YP Ma Physical Review A, Vol. 90 (2014) 023813

13. Passive mode-locking under higher order effects, T.P. Horikis and M.J. Ablowitz, J. Opt. Soc. Am. B, Vol. 31 (2014) 2748-2753

14. Integrable discrete PT symmetric model, M. J. Ablowitz and Z. H. Musslimani, Phys. Rev. E, Vol. 90 (2014) 032912

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Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, $K)

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